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# Realization of Robertson–Walker spacetimes as affine hypersurfaces

**Bang-Yen Chen**

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

E-mail: bychen@math.msu.edu

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**Abstract**
Due to the growing interest in embeddings of spacetimes in higher dimensional spaces, we consider a special type of embedding. We prove that Robertson–Walker spacetimes can be embedded as centroaffine hypersurfaces and graph hypersurfaces in some affine spaces in such a way that the induced relative metrics are exactly the Lorentzian metrics on the Robertson–Walker spacetimes. Such realizations allow us to view Robertson–Walker spacetimes and their submanifolds as affine submanifolds in a natural way. Consequently, our realizations make it possible to apply the tools of affine differential geometry to study Robertson–Walker spacetimes and their submanifolds.

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## 1. Introduction

One very important family of cosmological models in general relativity is the family of Robertson–Walker spacetimes:

$$M^n(k, f) := \left(I \times S, g_f^k\right), \qquad g_f^k = -\mathrm{d}t^2 + f^2(t) g_k, \tag{1.1}$$

equipped with a warped product Lorentzian metric $g_f^k$, where $f$ is a positive function defined on an open interval $I$ and $(S, g_k)$ is a Riemannian $(n-1)$-manifold of constant curvature $k = -1, 0$ or $-1$. The family of Robertson–Walker spacetimes includes the de Sitter, Minkowski and anti de Sitter spacetimes.

Robertson–Walker spacetimes are considered to be good descriptions of our Universe, except in the earliest era and the final era. A special case gives the Friedmann cosmological models (cf [7, 11]).

When $n = 2$, one may choose $S$ to be the real line; so it gives rise to a Robertson–Walker spacetime with the Lorentzian metric $g = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}s^2$. For $n \geqslant 3$, the standard choices for $S$ are the $(n-1)$-sphere $S^{n-1}$, the Euclidean space $\mathbb{E}^{n-1}$ and the hyperbolic $(n-1)$-space $H^{n-1}$, with curvature $+1, 0, -1$, respectively.

In recent years, the ideas of Theodor Kaluza and Oskar Klein have received new attention. Shortly after the publication of the theory of general relativity, Kaluza noted in April 1919 that when he solved Einstein's equations for general relativity using five dimensions, Maxwell's equations for electromagnetism emerged spontaneously. Kaluza wrote to Albert Einstein who encouraged him to publish. His theory was published in 1921 in [8] with Einstein's support. In 1926 Klein [9] suggested that this fifth dimension would be compactified and unobservable on experimentally accessible energy scales. However, their work was neglected for many years as attention was directed towards quantum mechanics. The idea that fundamental forces can be explained by additional dimensions did not re-emerge until string theory was developed. This idea of compactifying the extra dimension has now dominated the search for a unified theory and lead to the 11D supergravity theory and more recently the 10D superstring theory (see [12] for an overview). Recently, this strategy of using higher dimensions to unify different forces is also an active area of research in particle physics.

Instead of compactifying the extra dimensions, other approaches have also been developed. For example, one particular variant of the Kaluza–Klein (KK) theory is spacetime-matter (STM) theory or induced matter theory, chiefly promulgated by Paul Wesson and other members of the so-called Space-Time-Matter Consortium. In this version of the theory, it is noted that solutions to the equation $R_{AB} = 0$ with $R_{AB}$ as the 5D Ricci curvature may be re-expressed so that in four dimensions, these solutions satisfy Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ with the precise form of $T_{\mu\nu}$ following from the Ricci-flat condition on the 5D space. Since the energy–momentum tensor $T_{\mu\nu}$ is normally understood to be due to concentrations of matter in 4D space, the above result can be interpreted as saying that 4D matter is induced from geometry in a Ricci-flat 5D space. In particular, the soliton solutions of $R_{AB} = 0$ can be shown to contain the Robertson–Walker metric in both matter-dominated (early universe) and radiation-dominated (present universe) forms. The general equations can be shown to be sufficiently consistent with classical tests of general relativity to be acceptable on physical principles, while still leaving considerable freedom to also provide interesting cosmological models (see [17, 18]).

There is another approach proposed in 1999 by Lisa Randall and Raman Sundrum. Randall–Sundrum models imagine our Universe as a 5D anti de Sitter space, and the elementary particles except for the graviton are localized on a (3 + 1)-D brane or branes. Their models attempt to address the hierarchy problem between the observed Planck and weak scales by embedding the 3-brane in a warped 5D metric; the warping of the extra dimension is analogous to the warping of spacetime in the vicinity of a massive object, such as a black hole (see [13, 14] for details). The Randall–Sundrum scenario has gained a lot of support recently.

More recently, Stefan Haesen and Leopold Verstraelen investigated in [6] the embedding problem of spacetimes from the view point of ideal embeddings. The concept of ideal embeddings was originally introduced in 1990s by the author using author's $\delta$-invariants (see [2, 3, 6] for details). Roughly speaking, an ideal embedding is an isometric embedding which produces the least possible amount of tension from the ambient space at each point on the submanifold. Among others, Haesen and Verstraelen show in [6] that the 4D de Sitter and Robertson–Walker spacetimes can be ideally embedded in a 5D flat space.

In this paper, we investigate the embedding problem of spacetimes from the view point of affine differential geometry. More precisely, by applying an idea from [4] we prove that Robertson–Walker spacetimes $M$ can be realized as graph hypersurfaces and centroaffine hypersurfaces in some affine space in such a way that the induced fundamental form (i.e. the relative metric) is exactly the Lorentzian metric of $M$. Such realizations allow us to view

Robertson–Walker spacetimes and their submanifolds in a natural way as affine submanifolds in an affine space. Consequently, our realizations make it possible to apply the tools of affine differential geometry to study Robertson–Walker spacetimes and their submanifolds.

## 2. Affine hypersurfaces

In this section, we recall some well-known facts on affine hypersurfaces from affine differential geometry (see, for instance, [3, 10, 16] for details).

Let $\mathbf{R}^m$ denote the standard real $m$-dimensional vector space. For an ordered pair $(p, q)$ of points in $\mathbf{R}^m$, define $x = \overrightarrow{pq}$ to be the vector $q - p \in \mathbf{R}^m$. The vector space $\mathbf{R}^m$ together with the mapping

$$\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m : (p, q) \mapsto \overrightarrow{pq}$$

is called an affine $m$-space.

Let $Y$ be a vector field on $\mathbf{R}^m$ and let $x_t$, $a \leqslant t \leqslant b$, be an arbitrary smooth curve in $\mathbf{R}^m$. If we take an affine coordinate system $\{x^1, \ldots, x^m\}$ and write

$$Y = \sum_{i=1}^{m} Y^i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad x_t = (x^1(t), \ldots, x^m(t)),$$

then the covariant derivative $D_t Y$ of $Y$ along the curve $x_t$ is given by

$$D_t Y = \sum_{i=1}^{m} \frac{\mathrm{d}Y^i(x^i)}{\mathrm{d}t} \frac{\partial}{\partial x^i} = \sum_{i,j=1}^{m} \frac{\partial Y^i}{\partial x_j} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\partial}{\partial x^i}.$$

Thus, $D_t Y$ is a generalization of the directional derivative of functions to vector fields. If $X$ is a tangent vector at a point $x_0$, then $D_X Y$ is defined by $D_X Y = (D_t Y)_t$, where $x_t$ is a curve with the initial point $x_0$ and initial tangent vector $X$. From this definition, it is clear that the covariant differentiation $D$ has the following properties:

(1) $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y$,
(2) $D_{fX} Y = f D_X Y$,
(3) $D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2$,
(4) $D_X(fY) = (Xf)Y + f D_X Y$,

where $f$ is a smooth function and $X, Y, X_1, X_2, Y_1$ and $Y_2$ are vector fields.

By definition, an *affine connection* is a covariant differentiation on a smooth manifold satisfying the properties (1)–(4) given above.

Consider an affine $m$-space $\mathbf{R}^m$ equipped with the usual affine connection $D$ defined above. It is easy to verify that the curvature tensor associated with $D$ vanishes identically. So, the usual affine connection $D$ on $\mathbf{R}^m$ is a flat affine connection.

An immersion $\phi : M \to \mathbf{R}^{n+k}$ of an $n$-dimensional manifold $M$ into $\mathbf{R}^{n+k}$ is called an *affine immersion* if there exists a $k$-dimensional distribution $N$ on $M$ with $N_x \subset T_{\phi(x)}\mathbf{R}^{n+k}$ for each $x \in M$ such that

$$T_{\phi(x)}\mathbf{R}^{n+k} = \phi_*(T_x M) + N_x \quad \text{(direct sum)}, \tag{2.1}$$

$$(D_X \phi_*(Y))_x = (\phi_*(\nabla_X Y))_x + (\alpha(X, Y))_x, \tag{2.2}$$

at each point $x \in M$ for vector fields $X, Y$ on $M$, where $\phi_*$ is the differential of $\phi$ and $(\alpha(X, Y))_x$ denotes the $N_x$-component of $(D_X \phi_*(Y))_x$ according to the direct sum (2.1). Thus, (2.2) decomposes $D_X \phi_*(Y)$ into the tangential component $(\phi_*(\nabla_X Y))_x$ and the component $(\alpha(X, Y))_x$ in $N_x$ according to (2.1).

By an *affine hypersurface* $M$ in $\mathbf{R}^{n+1}$ we mean a codimension-1 affine immersion $\phi : M \to \mathbf{R}^{n+1}$ with a transversal vector field $\xi$, i.e. $\xi$ is a vector field which is not tangent to $M$ at each point on $M$. The formulae of Gauss and Weingarten for an affine hypersurface $\phi : M \to \mathbf{R}^{n+1}$ are given by

$$D_X\phi_*(Y) = \phi_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.3}$$

$$D_X\xi = -\phi_*(SX) + \tau(X)\xi, \tag{2.4}$$

where $X, Y$ are tangent vector fields on $M$ and $\phi_*(\nabla_X Y)$ and $-\phi_*(SX)$ are the tangential components of $D_X\phi_*(Y)$ and $D_X\xi$, respectively. It is known that $\nabla$ in (2.3) (induced from $D$ via $\phi$ according to (2.3)) is an affine connection, $h$ is a symmetric $(0, 2)$-tensor, $S$ is a (1,1)-tensor and $\tau$ is a 1-form on $M$.

The affine connection $\nabla$, the tensor $S$, the 1-form $\tau$ and the symmetric tensor $h$ are called the induced connection, the affine shape operator, the torsion form and the affine fundamental form (relative to the transversal vector field $\xi$), respectively.

An affine hypersurface $\phi : M \to \mathbf{R}^{n+1}$ is called *centroaffine* if the position vector field (from the origin $o$) is always transversal to the tangent hyperplane $\phi_*(T_x M)$ in $\mathbf{R}^{n+1}$ for each $x \in M$. For a centroaffine hypersurface, we always choose the transversal vector field $\xi$ to be $\phi$ (or more precisely, $\xi$ is the position vector field defined by $\phi(x) = \overrightarrow{ox}$ for $x \in M$). An affine hypersurface is called a *graph hypersurface* if the transversal vector field $\xi$ is a constant vector field.

When the induced fundamental form $h$ is nondegenerate for an affine hypersurface $\phi : M \to \mathbf{R}^{n+1}$, it defines a semi-Riemannian metric on $M$, called the *relative metric* (by 'relative' we mean 'relative with respect to the choice of $\xi$'). For graph hypersurfaces, the relative metric $h$ is also known as the *Calabi metric*.

In affine differential geometry, the transversal vector field $\xi$ of $\phi : M \to \mathbf{R}^{n+1}$ is called a *relative normalization* (or *equi-affine*) if $\tau = 0$ holds identically (i.e. $D_X\xi$ is tangent to $M$ for each $X \in TM$). The transversal vector field $\xi$ of a centroaffine hypersurface and that of a graph hypersurface are relative normalizations.

A Lorentzian manifold $M$ is said to be realized as an affine hypersurface if there exists a codimension-1 affine immersion of $M$ into an affine space such that the induced affine fundamental form $h$ (i.e. the relative metric) on $M$ is exactly the Lorentzian metric $g$ on $M$.

## 3. Robertson–Walker spacetimes as centroaffine hypersurfaces

Let $\mathbb{E}^{n-1}$, $S^{n-1}$ and $H^{n-1}$ denote the Euclidean $(n-1)$-space, $(n-1)$-sphere and hyperbolic $(n-1)$-space of constant sectional curvature 0, 1 and $-1$, respectively. With respect to a Euclidean coordinate system $\{u_2, \ldots, u_n\}$ on $\mathbb{E}^{n-1}$, the metric tensor $g_0$ on $\mathbb{E}^{n-1}$ is given by

$$g_0 = \sum_{j=2}^{n} \mathrm{d}u_j^2. \tag{3.1}$$

With respect to a spherical coordinate system $\{u_2, \ldots, u_n\}$ on $S^{n-1}$, the metric tensor $g_1$ on $S^{n-1}$ is

$$g_1 = \mathrm{d}u_2^2 + \cos^2 u_2\, \mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 u_j\, \mathrm{d}u_n^2. \tag{3.2}$$

Similarly, for $H^{n-1}$, the corresponding metric tensor $g_{-1}$ on $H^{n-1}$ is

$$g_{-1} = \mathrm{d}u_2^2 + \cosh^2 u_2\, \mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1} \cosh^2 u_j\, \mathrm{d}u_n^2. \tag{3.3}$$

We have the following realization theorem.

**Theorem 1.** *Let $f$ be a positive function defined on an open interval $I \ni 0$. We have*

*(i) every Robertson–Walker spacetime $M^2(0, f)$ can always be realized as centroaffine hypersurfaces in $\mathbf{R}^3$;*
*(ii) for any integer $n \geqslant 2$, the Robertson–Walker spacetime $M^n(1, f)$ can always be realized as centroaffine hypersurfaces in $\mathbf{R}^{n+1}$;*
*(iii) if $f > 1$ holds on $I$, then the Robertson–Walker spacetime $M^n(-1, f)$ can be realized as a centroaffine hypersurface in $\mathbf{R}^{n+1}$.*

**Proof.** (a) It follows from (3.1) that the metric tensor $g$ of the Robertson–Walker spacetime $M^2(0, f) = I \times_f \mathbb{E}^1$ is given by the warped product Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}u^2. \tag{3.4}$$

Let us consider the embedding $\phi : M^2(0, f) \to \mathbf{R}^3$ defined by

$$\phi = \left( \exp\left\{ u + \int_0^t \frac{f' + \sqrt{f'^2 + f^2 + 1}}{f}\,\mathrm{d}t \right\},\right.$$
$$\left. u\exp\left\{ u + \int_0^t \frac{f' + \sqrt{f'^2 + f^2 + 1}}{f}\,\mathrm{d}t \right\}, \exp\left\{ \int_0^t \frac{f(f' + \sqrt{f'^2 + f^2 + 1})}{f^2 + 1}\,\mathrm{d}t \right\} \right) \tag{3.5}$$

It is easy to verify that the embedding $\phi$ satisfies

$$\begin{cases} \phi_{tt} = \dfrac{1 + 2f^2 + f'^2 + ff''}{f\sqrt{1 + f^2 + f'^2}}\phi_t - \phi, \\ \phi_{tu} = \dfrac{f' + \sqrt{1 + f^2 + f'^2}}{f}\phi_u, \\ \phi_{uu} = 2\phi_u - \dfrac{f(1 + f^2)}{f' + \sqrt{1 + f^2 + f'^2}}\phi_t + f^2(t)\phi, \end{cases} \tag{3.6}$$

where $\phi_t = \phi_*\left(\frac{\partial}{\partial t}\right)$, $\phi_u = \phi_*\left(\frac{\partial}{\partial u}\right)$ are tangent vector fields of $M$.

Let us choose the transversal vector field $\xi$ to be $\phi$. Then it follows from equations (2.3) and (3.6) that the induced affine fundamental form $h$ on the Robertson–Walker spacetime $M^2(0, f)$ is given by (this is done by comparing the component of $\xi$ from equation (2.3) and of $\phi$ from equation (3.6))

$$h = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}u^2, \tag{3.7}$$

which is exactly the Lorentzian metric (3.4) on $M^2(0, f)$. Hence, the embedding defined by (3.5) is a realization of $M^2(0, f)$ as a centroaffine hypersurface. This proves statement (i) of the theorem.

(b) Consider the Robertson–Walker spacetime $M^n(1, f) = I \times_f S^{n-1}$ equipped with the warped product Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\left\{ \mathrm{d}u_2^2 + \cos^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cos^2 u_j\,\mathrm{d}u_n^2 \right\}. \tag{3.8}$$

Define $\eta : M^n(1, f) \to \mathbf{R}^{n+1}$ by

$$\eta(t, u_2, \ldots, u_n) = \left(0, \ldots, 0, \exp\left(\int_0^t \frac{f(f' + \sqrt{f'^2 + f^2 + 1})}{f^2 + 1}\,\mathrm{d}t\right)\right) + f(t)\exp\left(\int_0^t \frac{\sqrt{f'^2 + f^2 + 1}}{f}\,\mathrm{d}t\right)\left(\sin u_2, \ldots, \sin u_n \prod_{j=2}^{n-1}\cos u_j, \prod_{j=2}^{n}\cos u_j, 0\right). \tag{3.9}$$

By a long straightforward computation, we obtain

$$\begin{cases} \eta_{tt} = \dfrac{f'^2 + ff'' + 2f^2 + 1}{f\sqrt{f'^2 + f^2 + 1}}\eta_t - \eta, \\ \eta_{tu_j} = \dfrac{f' + \sqrt{f'^2 + f^2 + 1}}{f}\eta_{u_j}, \qquad j = 2, \ldots, n, \\ \eta_{u_iu_j} = -(\tan u_i)\eta_{u_j}, \qquad 2 \leqslant i < j \leqslant n, \\ \eta_{u_ju_j} = f^2\left(\displaystyle\prod_{i=2}^{j-1}\cos^2 u_i\right)\eta - \dfrac{f + f^3}{f' + \sqrt{f'^2 + f^2 + 1}}\left(\displaystyle\prod_{i=2}^{j-1}\cos^2 u_i\right)\eta_t \\ \quad + \displaystyle\sum_{k=2}^{j-1}\left(\frac{\sin(2u_k)}{2}\prod_{i=k+1}^{j-1}\cos^2 u_i\right)\eta_{u_k}, \qquad 2 \leqslant j \leqslant n. \end{cases} \tag{3.10}$$

Comparing equation (2.3) with equation (3.10) in the same way as above shows that, with $\xi = \eta$, the induced affine fundamental form on the Robertson–Walker spacetime $M^n(1, f)$ is given by

$$h = -\mathrm{d}t^2 + f^2(t)\left\{\mathrm{d}u_2^2 + \cos^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cos^2 u_j\,\mathrm{d}u_n^2\right\}, \tag{3.11}$$

which is exactly the Lorentzian metric (3.8) on $M^n(1, f)$. Consequently, (3.9) gives a realization of $M^n(1, f)$ as a centroaffine hypersurface. This proves statement (ii).

(c) Assume that $f > 1$ holds on $I$. Consider the Robertson–Walker spacetime $M^n(-1, f) = I \times_f H^{n-1}$ with the warped product Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\left\{\mathrm{d}u_2^2 + \cosh^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cosh^2 u_j\,\mathrm{d}u_n^2\right\}. \tag{3.12}$$

Define $\psi : M^n(-1, f) \to \mathbf{R}^{n+1}$ by

$$\psi = \left(0, \ldots, 0, \exp\left(\int_0^t \frac{f(f' + \sqrt{f'^2 + f^2 - 1})}{f^2 - 1}\,\mathrm{d}t\right)\right) + f(t)\exp\left(\int_0^t \frac{\sqrt{f'^2 + f^2 - 1}}{f}\,\mathrm{d}t\right) \times \left(\sinh u_2, \ldots, \sinh u_n \prod_{j=2}^{n-1}\cosh u_j, \prod_{j=2}^{n}\cosh u_j, 0\right). \tag{3.13}$$

A long straightforward computation implies that

$$\begin{cases} \psi_{tt} = \dfrac{f'^2 + ff'' + 2f^2 - 1}{f\sqrt{f'^2 + f^2 - 1}}\psi_t - \psi, \\ \psi_{tu_j} = \dfrac{f' + \sqrt{f'^2 + f^2 - 1}}{f}\psi_{u_j}, \qquad j = 2, \ldots, n, \\ \psi_{u_i u_j} = (\tanh u_i)\psi_{u_j}, \qquad 2 \leqslant i < j \leqslant n, \\ \psi_{u_j u_j} = \left(f^2\psi + \dfrac{f - f^3}{f' + \sqrt{f'^2 + f^2 - 1}}\psi_t\right)\displaystyle\prod_{i=2}^{j-1}\cosh^2 u_i \\ \quad -\displaystyle\sum_{k=2}^{j-1}\left(\dfrac{\sinh(2u_k)}{2}\prod_{i=k+1}^{j-1}\cosh^2 u_i\right)\psi_{u_k}, \qquad j = 2, \ldots, n. \end{cases} \tag{3.14}$$

Comparing equation (2.3) with equation (3.14) in a same way as before shows that, with $\xi = \psi$, the induced affine fundamental form on $M^n(-1, f)$ is given by

$$h = -\mathrm{d}t^2 + f^2(t)\left\{\mathrm{d}u_2^2 + \cosh^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cosh^2 u_j\,\mathrm{d}u_n^2\right\}, \tag{3.15}$$

which is exactly the Lorentzian metric on $M^n(-1, f)$. Hence, we have a realization of the Robertson–Walker spacetime $M^n(-1, f)$ as a centroaffine hypersurface. This proves statement (iii). □

## 4. Robertson–Walker spacetimes as graph hypersurfaces

First, we observe that the Minkowski spacetime $\mathbb{E}_1^n$, $n \geqslant 2$, with the canonical Lorentzian metric

$$g = -\mathrm{d}t^2 + \sum_{j=2}^{n}\mathrm{d}u_j^2 \tag{4.1}$$

can be realized as a graph hypersurface in an affine $(n+1)$-space $\mathbf{R}^{n+1}$. This can be easily done as follows. Consider the embedding $\phi : \mathbb{E}_1^n \to \mathbf{R}^{n+1}$ defined by

$$\phi(t, u_2, \ldots, u_n) = \left(t, u_2, \ldots, u_n, -\frac{t^2}{2} + \frac{1}{2}\sum_{j=2}^{n}u_j^2\right). \tag{4.2}$$

It is easy to verify that the embedding satisfies

$$\phi_{tt} = -\xi, \qquad \phi_{u_j u_k} = \delta_{jk}\xi, \qquad \phi_{tu_j} = 0, \tag{4.3}$$

where $\xi$ is the constant transversal vector field given by $\xi = (0, \ldots, 0, 1)$. It follows from (2.3) and (4.3) that the induced Calabi metric $h$ on the Minkowski spacetime $\mathbb{E}_1^n$ via $\phi$ is exactly the Minkowski metric (4.1) on $\mathbb{E}_1^n$. Therefore, (4.2) gives rise to a realization of $\mathbb{E}_1^n$ in $\mathbf{R}^{n+1}$ as a graph hypersurface.

For Robertson–Walker spacetimes, we also have the following realization theorem.

**Theorem 2.** *Let $f$ be a positive function defined on an open interval $I$. We have*

*(a) every 2D Robertson–Walker spacetime $M^2(0, f) = I \times_f \mathbf{R}$ can be realized as a graph surface in an affine 3-space $\mathbf{R}^3$;*
*(b) for $n \geqslant 3$, we have the following.*

*(b.1) The Robertson–Walker spacetime* $M^n(1, f) = I \times_f S^{n-1}$ *can always be realized as a graph hypersurface in* $\mathbf{R}^{n+1}$*;*

*(b.2) If* $f$ *has no critical points in* $I$*, then the Robertson–Walker spacetime* $M^n(0, f) = I \times_f \mathbb{E}^{n-1}$ *can be realized as a graph hypersurface in* $\mathbf{R}^{n+1}$*;*

*(b.3) If* $f'^2(t) > 1$ *holds for each* $t \in I$*, then the Robertson–Walker spacetime* $M^n(-1, f) = I \times_f H^{n-1}$ *can be realized as a graph hypersurface in* $\mathbf{R}^{n+1}$.

**Proof.** Without loss of generality, we may assume that the interval $I$ contains 0.

(A) Consider the 2D Robertson–Walker spacetime equipped with the warped product Lorentzian metric: $g = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}u^2$. Let us put

$$\psi(t,u) = \left( f(t)\exp\left\{\int_0^t \frac{\sqrt{f'^2+1}}{f}\,\mathrm{d}t\right\}\sin u,\ f(t)\exp\left\{\int_0^t \frac{\sqrt{f'^2+1}}{f}\,\mathrm{d}t\right\}\cos u, \right.$$
$$\left. \times \int_0^t f(f' + \sqrt{f'^2+1})\,\mathrm{d}t \right). \tag{4.4}$$

Then a direct computation shows that $\psi$ satisfies

$$\begin{cases} \psi_{tt} = \dfrac{ff'' + f'^2 + 1}{f\sqrt{f'^2+1}}\psi_t - \xi, \\ \psi_{tu} = \dfrac{f' + \sqrt{f'^2+1}}{f}\psi_u, \\ \psi_{uu} = f(f' - \sqrt{f'^2+1})\psi_t + f^2\xi \end{cases} \tag{4.5}$$

with $\xi = (0, 0, 1)$.

It follows from (2.3) and (4.5) that the induced Calabi metric $h$ is exactly the Lorentzian metric $g = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}u^2$. This shows that every 2D Robertson–Walker spacetime can be realized as a graph surface in $\mathbf{R}^3$. This proves statement (a) of the theorem.

(B) Consider the Robertson–Walker spacetime $M^n(1, f) = I \times_f S^{n-1}$ equipped with Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\left\{\mathrm{d}u_2^2 + \cos^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cos^2 u_j\,\mathrm{d}u_n^2\right\}. \tag{4.6}$$

Define $\eta : M^n(1, f) \to \mathbf{R}^{n+1}$ by

$$\eta(t, u_2, \ldots, u_n) = \left(0, \ldots, 0, \int_0^t f(f' - \sqrt{f'^2+1})\,\mathrm{d}t\right) + f\exp\left(-\int_0^t \{\sqrt{f'^2+1}/f\}\,\mathrm{d}t\right)$$
$$\times \left(\sin u_2, \sin u_3\cos u_2, \ldots, \sin u_n\prod_{j=2}^{n-1}\cos u_j, \prod_{j=2}^{n}\cos u_j, 0\right).$$

A straightforward long computation yields

$$\begin{cases} \eta_{tt} = -\dfrac{1+ff''+f'^2}{f\sqrt{f'^2+1}}\eta_t - \xi, \\ \eta_{tu_j} = \dfrac{f'-\sqrt{f'^2+1}}{f}\eta_{u_j}, \qquad j=2,\ldots,n, \\ \eta_{u_iu_j} = -(\tan u_i)\eta_{u_j}, \qquad 2 \leqslant i < j \leqslant n, \\ \eta_{u_ju_j} = \left\{\dfrac{f}{\sqrt{f'^2+1}-f'}\eta_t + f^2\xi\right\}\displaystyle\prod_{i=2}^{j-1}\cos^2 u_i \\ \quad + \displaystyle\sum_{k=2}^{j-1}\left(\frac{\sin 2u_k}{2}\prod_{i=k+1}^{j-1}\cos^2 u_i\right)\eta_{u_k}, \qquad j=2,\ldots,n \end{cases} \tag{4.7}$$

with $\xi = (0,\ldots,0,1)$. It follows from (2.3) and (4.7) that the induced Calabi metric on the Robertson–Walker spacetime $M^n(1,f)$ via $\eta$ is exactly the Lorentzian metric (4.6) on $M^n(1,f)$. Thus, $\eta$ is a realization of $M^n(1,f)$. This proves statement (b.1).

Now, assume that the warping function $f$ does not have critical points. Consider the Robertson–Walker spacetime $M(0,f) = I \times_f \mathbb{E}^{n-1}$ with the Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\left(\mathrm{d}u_2^2 + \mathrm{d}u_3^2 + \cdots + \mathrm{d}u_n^2\right). \tag{4.8}$$

Define $\phi : M(0,f) \to \mathbf{R}^{n+1}$ by

$$\zeta(t,u_2,\ldots,u_n) = \left(u_2,\ldots,u_n, \sum_{j=2}^{n} u_j^2 + \int_0^t \frac{\mathrm{d}t}{f(t)f'(t)}, \int_0^t \frac{f(t)}{2f'(t)}\,\mathrm{d}t\right). \tag{4.9}$$

Then a straightforward computation yields

$$\begin{cases} \zeta_{tt} = -\dfrac{f'^2+ff''}{ff'}\zeta_t - \xi, \\ \zeta_{tu_j} = \zeta_{u_iu_j} = 0, \qquad 2 \leqslant i \neq j \leqslant n, \\ \zeta_{u_ju_j} = 2ff'\zeta_t + f^2\xi, \qquad 2 \leqslant j \leqslant n, \end{cases} \tag{4.10}$$

where $\xi = (0,\ldots,0,1)$ is a constant transversal vector field. It follows from (2.3) and (4.9) that the induced Calabi metric $h$ via $\zeta$ is exactly the Lorentzian metric (4.8) on $M^n(0,f)$. Thus, $\zeta$ is a realization of $M^n(0,f)$ as a graph hypersurface. This proves statement (b.2)

Finally, assume that $f'^2(t) > 1$ holds for each $t \in I$. Consider the Robertson–Walker spacetime $M^n(-1,f) = I \times_f H^{n-1}$ with the Lorentzian metric:

$$g = -\mathrm{d}t^2 + f^2(t)\left\{\mathrm{d}u_2^2 + \cosh^2 u_2\,\mathrm{d}u_3^2 + \cdots + \prod_{j=2}^{n-1}\cosh^2 u_j\,\mathrm{d}u_n^2\right\}. \tag{4.11}$$

Define $\phi : M^n(-1,f) \to \mathbf{R}^{n+1}$ by

$$\phi(t,u_2,\ldots,u_n) = \left(0,\ldots,0, \int_0^t f\left(\sqrt{f'^2-1} - f'\right)\mathrm{d}t\right) + \frac{f}{\mathrm{e}^{\int_0^t\{\sqrt{f'^2-1}/f\}\,\mathrm{d}t}}$$
$$\times\left(\sinh u_2, \sinh u_3\cosh u_2, \ldots, \sinh u_n\prod_{j=2}^{n-1}\cosh u_j, \prod_{j=2}^{n}\cosh u_j, 0\right).$$

Then a straightforward long computation yields

$$\begin{cases} \phi_{tt} = \dfrac{1 - ff'' - f'^2}{f\sqrt{f'^2 - 1}}\phi_t - \xi, \\ \phi_{tu_j} = \dfrac{f' - \sqrt{f'^2 - 1}}{f}\phi_{u_j}, \qquad j = 2, \ldots, n, \\ \phi_{u_i u_j} = (\tanh u_i)\phi_{u_j}, \qquad 2 \leqslant i < j \leqslant n, \\ \phi_{u_j u_j} = \left\{ \dfrac{f}{f' - \sqrt{f'^2 - 1}}\phi_t + f^2\xi \right\} \displaystyle\prod_{i=2}^{j-1} \cosh^2 u_i \\ \quad - \displaystyle\sum_{k=2}^{j-1} \left\{ \dfrac{\sinh 2u_k}{2} \prod_{i=k+1}^{j-1} \cosh^2 u_i \right\} \phi_{u_k}, \qquad j = 2, \ldots, n \end{cases} \tag{4.12}$$

with $\xi = (0, \ldots, 0, 1)$. It follows from (2.3) and (4.12) that the induced Calabi metric $h$ on the Robertson–Walker spacetime $M^n(-1, f)$ via $\phi$ is exactly the Lorentzian metric (4.11) on $M^n(-1, f)$. Hence, $\phi$ gives a realization of $M^n(-1, f)$ as a graph hypersurface. This proves statement (b.3). □

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